Critical behavior of the two-dimensional N-component Landau-Ginzburg Hamiltonian with cubic anisotropy.

Pasquale Calabrese [1] and Alessio Celi [2]

[1] *Scuola Normale Superiore and INFN, Piazza dei Cavalieri 7, I-56126 Pisa, Italy.*
[2] *Dipartimento di Fisica dell'Università di Milano and INFN, Via Celoria 16, I-20133 Milano, Italy.*
e-mail: calabres@df.unipi.it, celi@mi.infn.it
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We study the two-dimensional N-component Landau-Ginzburg Hamiltonian with cubic anisotropy. We compute and analyze the fixed-dimension perturbative expansion of the renormalization-group functions to four loops. The relations of these models with N-color Ashkin-Teller models, discrete cubic models, planar model with fourth order anisotropy, and structural phase transition in adsorbed monolayers are discussed. Our results for $N = 2$ (XY model with cubic anisotropy) are compatible with the existence of a line of fixed points joining the Ising and the $O(2)$ fixed points. Along this line the exponent η has the constant value $1/4$, while the exponent ν runs in a continuous and monotonic way from 1 to ∞ (from Ising to $O(2)$). For $N \geq 3$ we find a cubic fixed point in the region $u, v \geq 0$, which is marginally stable or unstable according to the sign of the perturbation. For the physical relevant case of $N = 3$ we find the exponents $\eta = 0.17(8)$ and $\nu = 1.3(3)$ at the cubic transition.

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I. INTRODUCTION

In the framework of Renormalization Group (RG) approach to critical phenomena, the critical behavior at many continuous phase transitions can be investigated by considering an effective Landau-Ginzburg Hamiltonian, having an N-component order parameter ϕ_i as fundamental field, and containing up to fourth-order powers of the field components. The fourth-degree polynomial form of the potential depends essentially on the symmetry of the system. In fact according to the universality hypothesis, the critical properties of these systems can be described in terms of quantities that do not depend on the microscopic details of the system, but only on global properties such as the dimensionality and the symmetry of the order parameter, and the range of the interactions.

The critical properties of many magnetic materials are computed using the $O(N)$-invariant Landau-Ginzburg Hamiltonian. Uniaxial ferromagnets should be described by the Ising universality class ($N = 1$), while magnets with easy-plane anisotropy should belong to the XY universality class. Ferromagnets are often described in terms of the $O(3)$ Hamiltonian. However, this is correct if the non-rotationally invariant interactions that have only the reduced symmetry of the lattice are irrelevant in the renormalization-group sense. In two dimensions the effect of anisotropy is very important: systems possessing continuous symmetry do not exhibit conventional long-range order at finite temperature, while models with discrete symmetry do undergo phase transitions into conventionally ordered phase.

For studying the effect of cubic anisotropies one usually consider the ϕ^4 theory [1,2]:

$$\mathcal{H} = \int d^d x \left\{ \frac{1}{2} \sum_{i=1}^{N} \left[(\partial_\mu \phi_i)^2 + r\phi_i^2 \right] \right.$$

$$\left. + \frac{1}{4!} \sum_{i,j=1}^{N} \left(u_0 + v_0 \delta_{ij} \right) \phi_i^2 \phi_j^2 \right\}, \qquad (1)$$

in which the added cubic term breaks explicitly the $O(N)$ invariance leaving a residual discrete cubic symmetry given by the reflections and permutations of the field components [3]. This term favors the spin orientations towards the faces or the corners of an N-dimensional hypercube for $v_0 < 0$ or $v_0 > 0$ respectively.

The Hamiltonian (1) has received much attention also because it describes the two-dimensional order-disorder transition in some adsorbed monolayers. In fact, in the original classification [4] on square and rectangular lattices, all these transitions belong to the universality class of the Ising model, three- or four-state Potts model, and XY model with cubic anisotropy. In the successive extension [5] to non-Bravais lattices, as the honeycomb and the Kagomé ones, also the Heisenberg ($N = 3$) model with cubic anisotropy is interested by this classification (e.g. in the adsorption of diatomic-molecules on graphite). The full classification of continuous phase transition of magnetic symmetry in two dimensions [6] reveals the interest of the model (1) also for other kind of transitions.

An important application of the XY model with cubic anisotropy is in the oxygen ordering in $YBaCu_3O_{6+x}$ [7,8], since it is one of the most studied high-temperature superconductors. Some other applications of Landau-Ginzburg Hamiltonian (1) are: the buckling instabilities of a confined colloid crystal layer [9], some discrete models with competing nearest- and next-nearest-neighbor interactions [10], and, very recently [11], it is argued that the phase diagram of a lattice-gas model for studying the micellar binary solution of water and ampliphile is partially described by the XY model with cubic anisotropy. We also mention that, in the limit $N \to 0$, the cubic

model (1) describes the Ising model with site-diluted disorder [12–14], but we do not consider it here, since in the framework of fixed dimension $d = 2$ it was already analyzed in Ref. [15].

In this paper we present a field-theoretic study based on an expansion performed directly in two dimensions, as proposed for the $O(N)$ models by Parisi [16].

The paper is organized as follows. In Sec. II we give an overview of all known results that we believe necessary for a full understanding of the two-dimensional cubic model (1). In Sec. III we derive the perturbative series for the renormalization-group functions at four loops and discuss the singularities of the Borel transform. The results of the analysis are presented in Sec. IV. The reader that is not interested to the detail of the calculations can skip sections III and IV and read directly Sec. V where we summarize all our results and point out some questions which we think deserve further study.

II. OVERVIEW OF KNOWN RESULTS

A. Three dimensional results

We shortly review the most interesting features appearing in the study of the Hamiltonian (1) in the framework of ϵ-expansion [17,1] and at fixed dimension $d = 3$ [18], since they are necessary for a good understanding of the two-dimensional case.

The model (1) has four fixed points: the trivial Gaussian one, the Ising one in which the N components of the field decouple, the $O(N)$-symmetric and the cubic fixed points. The Gaussian fixed point is always unstable, and so is the Ising fixed point [19]. Indeed, in the latter case, it is natural to interpret Eq. (1) as the Hamiltonian of N Ising-like systems coupled by the $O(N)$-symmetric term. But this interaction is the sum of the products of the energy operators of the different Ising systems. Therefore, at the Ising fixed point, the crossover exponent associated to the $O(N)$-symmetric quartic term should be given by the specific-heat critical exponent α_I of the Ising model, independently of N. Since α_I is positive for all $d > 2$ the Ising fixed point is unstable. Obviously in two dimensions this argument fails since $\alpha_I = 0$.

While the Gaussian and the Ising fixed points are unstable for any number of components N, the stability properties of the $O(N)$-symmetric and of the cubic fixed points depend on N. For sufficiently small values of N, $N < N_c$, the $O(N)$-symmetric fixed point is stable and the cubic one is unstable. For $N > N_c$, the opposite is true: the renormalization-group flow is driven towards the cubic fixed point, which now describes the generic critical behavior of the system. The $O(N)$-symmetric point corresponds to a tricritical transition. Figure 1 sketches the flow diagram in the two cases $N < N_c$ and
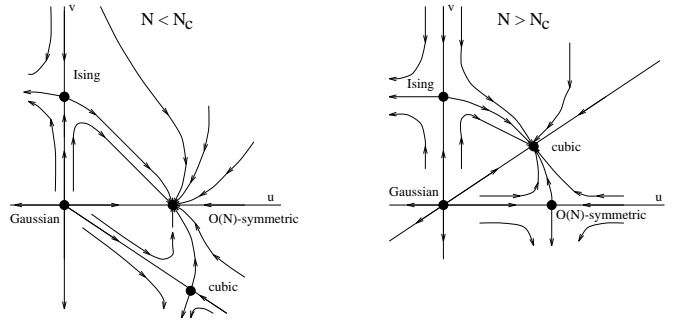

FIG. 1. Renormalization-group flow in the coupling plane (u, v) for $N < N_c$ and $N > N_c$.

$N > N_c$. At $N = N_c$, the two fixed points should coincide, and logarithmic corrections to the $O(N)$-symmetric critical exponents are expected. Outside the attraction domain of the fixed points, the flow goes away towards more negative values of u and/or v and finally reaches the region where the quartic interaction no longer satisfies the stability condition. These trajectories should be related to first-order phase transitions [20]. Some recent and very accurate calculations [21–27,18] suggest that in three-dimensions $N_c < 3$.

If $N > N_c$, the cubic anisotropy is relevant and therefore the critical behavior of the system is not described by the Heisenberg isotropic Hamiltonian. If the cubic interaction favors the alignment of the spins along the diagonals of the cube, i.e. for a positive coupling v_0, the critical behavior is controlled by the cubic fixed point and the cubic symmetry is retained even at the critical point. On the other hand, if the system tends to magnetize along the cubic axes — this corresponds to a negative coupling v_0 — then the system undergoes a first-order phase transition [1,28–30].

In the limit $N \to \infty$, keeping Nu and v fixed, one can derive exact expressions for the exponents at the cubic fixed point for all dimensions. Indeed, in this limit the model can be reinterpreted as a constrained Ising model [31], leading to a Fisher renormalization of the Ising critical exponents [32]. One has [33,31,1]:

$$\eta = \eta_I + O\left(\frac{1}{N}\right), \qquad \nu = \frac{\nu_I}{1 - \alpha_I} + O\left(\frac{1}{N}\right), \ (2)$$

where η_I, ν_I, and α_I are the critical exponents of the Ising model.

In all dimensions, a simple argument based on the symmetry of the two-component cubic model [34] shows that the cubic fixed point for $N = 2$ has the same stability properties of the Ising one. Indeed, for $N = 2$, a $\pi/4$ internal rotation, i.e.

$$(\phi_1, \phi_2) \longrightarrow \frac{1}{\sqrt{2}}(\phi_1 + \phi_2, \phi_1 - \phi_2), \qquad (3)$$

maps the cubic Hamiltonian (1) into a new one of the same form but with new couplings (u'_0, v'_0) given by

$$u'_0 = u_0 + \tfrac{3}{2}v_0, \qquad v'_0 = -v_0. \qquad (4)$$

This symmetry maps the Ising fixed point onto the cubic one. So for all $d > 2$, since the Ising point is unstable, the cubic point is unstable too, and the stable point is the isotropic one. In two dimensions, this is no longer true. Indeed, one expects the cubic interaction to be truly marginal for $N = 2$ [35,36] and relevant for $N > 2$ [37], so $N_c = 2$ in two dimensions.

B. The two dimensional case

As it is already clear from the previous subsection, in two-dimensions several new and interesting features appear. First of all we have no general argument to understand the stability properties of the Ising fixed point, in fact the specific heat of the two-dimensional Ising model has a logarithmic divergence (i.e. $\alpha_I = 0$). The vanishing of this crossover exponents, and so the presence of a marginal operator, can bring us to several different scenarios. For example we may have that the Ising fixed point is marginally stable or unstable because of higher order corrections to RG equations, or that there exists a line of fixed points (see Ref. [38] for a detailed review about the effects of marginal operators).

When the cubic anisotropy becomes marginal (at $N = N_c = 2$) [37] the cubic fixed point is in the region with $v < 0$ (cf. Eq. (4)), and so it cannot coincide with the $O(N)$ one, as in the case of three and $4 - \epsilon$ dimensions.

There are several studies on systems related to Landau-Ginzburg Hamiltonian (1). In Ref. [35] the model

$$\mathcal{H} = \sum_{\langle \mathbf{r}, \mathbf{r}' \rangle} J\, \mathbf{S}(\mathbf{r})\mathbf{S}(\mathbf{r}') + \sum_{\mathbf{r}} \mathbf{h_p} \cos \mathbf{p}\theta(\mathbf{r}) \qquad (5)$$

was considered, where the first sum is only over nearest-neighbors and $\theta(\mathbf{r})$ is the angle that the two-component spin $\mathbf{S}(\mathbf{r})$ forms with some arbitrary axis. This system clearly belongs to the same universality class of XY model with cubic anisotropy if $p = 4$ (the "field" h_4 maps on v_0).

The phase diagram found in Ref. [35] consists of three distinct lines of fixed points showing continuously varying exponents. The line with $h_4 = 0$ is the standard Kosterlitz-Thouless (KT) [39]. There are two lines with $h_4 \neq 0$, starting from the KT transition (the end point of the KT line) and continuing to infinitely, positive or negative, large values of h_4. They map into each other with changing the sign of h_4. The latter are lines of second-order phase transitions with conventional power law singularities characterized by $\eta = 1/4$. The exponent ν diverges at the confluence of these critical lines as

$$\nu \sim \frac{1}{|h_4|} \qquad (6)$$

when $h_4 \to 0$. For this reason it is often said that the XY model with cubic anisotropy has a non-universal behavior, although there is the so-called weak-universality of Suzuki [40] (i.e. η is constant).

In a successive work Kadanoff [41] identifies the two fixed-point lines for $h_4 \neq 0$ with the dual line of the Ashkin-Teller (AT) and the eight-vertex Baxter model (8V) (see the Baxter book [42] for a review about these models). One remark is necessary at this point: the AT and the 8V models are equivalent on the dual line [42] and they show a continuously varying exponent of the correlation length:

$$\nu = \frac{2 - y}{3 - 2y}, \qquad (7)$$

where y is a parameter that appears in the Hamiltonians defined in the range $[0, 4/3]$ for the AT [43] and $[0, 2]$ for the 8V [42]. The $O(2)$ multicritical point identified by Kadanoff is the F-model limit of the 8V that is characterized by $y = 3/2$, allowed to the 8V but forbidden for the AT.

Another class of models related to the Landau-Ginzburg Hamiltonian (1) is the discrete-cubic N-component model [28,44–48]. It is a short-range interacted system with N-component spins $\mathbf{S_i}$ pointing to the faces of an N-dimensional hypercube (face-centered-cubic model). The Hamiltonian may be written as:

$$\mathcal{H} = \sum_{\langle i,j \rangle} (C\delta_{a_i,a_j} s_i s_j + P\delta_{a_i,a_j}) \qquad (8)$$

where a_i is a Potts-like variable that determines which component of $\mathbf{S_i}$ is non-zero and s_i an Ising variable that determines the sign of that component. For $P = 0$ the Hamiltonian (8) reduces to the $2N$-state Potts model, for $C = 0$ to two decoupled N-state Potts model, and for $N = 2$ to the AT model. The continuous cubic model (1) reduces to the discrete one (8) in the limit of strong anisotropy ($|v| \gg |u|$). In Ref. [46] it is shown that the iteration of RG transformations enforces the continuous model, with $v_0 < 0$, to have spins pointing only to the faces of the hypercube, and so, also for finite anisotropy, the two models are equivalent at criticality. The model (8) exhibits four competing possible types of critical behavior, related to the Ising model, the N-and $2N$-state Potts models and to a "cubic" fixed point. In Refs. [47,48] it was found that the critical behavior of the discrete face-centered cubic model is $O(N)$-like for $N < N_c = 2$, AT-like for $N = 2$, and characterized by a first-order phase transition for $N > 2$. This result is not surprising since the model (8) is related to (1) in the region with $v_0 < 0$ where we expect a first-order phase transition for $N > N_c = 2$ and $O(N)$ behavior for $N < 2$.

In the region with $v_0 > 0$ the continuous model (1) is related to the corner-cubic model in which N-component spins $\mathbf{S_i}$ point to the corners of an N-dimensional hypercube. This model is equivalent to the face-centered one for $N = 2$ (this is the symmetry (4) for the Hamiltonian (1) and the changing of the sign of h_4 for (5)). For other values of N the critical behavior of the face-centered- and the corner-cubic model is in principle very different.

This difference is clarified by the study of N-color Ashkin-Teller model, first introduced by Grest and Widom [49], that is equivalent to the corner-cubic model. The Hamiltonian of this model reads

$$\mathcal{H} = -\sum_{\langle i,j \rangle} \left[J \sum_{a=1}^{N} s_i^a s_j^a + J_4 \left[\sum_{a=1}^{N} s_i^a s_j^a \right]^2 \right], \qquad (9)$$

where $s_i^a = \pm 1$ for $a = 1, \ldots, N$ are N Ising variables and the sum is only over nearest-neighbor (the sign of J_4 is the opposite of u_0 in Eq. (1)). In Ref. [49] it is concluded that the order J_4^2 in the RG equations makes the decoupled Ising fixed point stable for perturbations with $J_4 < 0$ ($u_0 > 0$) and unstable for $J_4 > 0$ ($u_0 < 0$). In the latter case the system flows away towards more negative values of u and finally the transition is first order. This has been confirmed [49] by a Monte Carlo simulation for $N = 3$. This argument was proved exactly in the $N \to \infty$ limit [50,51]. After, using a mapping of the Hamiltonian (9) onto the $O(N)$ Gross-Neveu model [52] it was argued [52–55] that the critical behavior of the Ising fixed point is affected by logarithmic corrections.

Finally the three couplings $(N,2)$ model of Domany and Riedel [46] (equivalent for $N = 3$ to the $Z(6)$ model [56]) reduces to the N-component discrete cubic model, for particular values of the parameters entering in the Hamiltonian. The model $(3,2)$ was studied in Refs. [46,56] with the Migdal-Kadanoff Renormalization Group approach (MKRG). Unfortunately the MKRG is expected to give neither precise values of the exponents nor the correct nature of the transition, but only a good description of the phase diagram.

C. Order-disorder transition in adsorbed monolayers

We have already mentioned that the Hamiltonian (1) describes some order-disorder transitions in adsorbed monolayers. The great interest in these models is justified since they provide a unique possibility to study experimentally a rich variety of two-dimensional systems.

Several experimental works have confirmed that these transitions belong to the universality class of the Ising [57–59], the three- [60,61] and four-state Potts models [62,61], and the XY model with cubic anisotropy [63–66], according to their classification [4–6]. Monte Carlo simulations and other numerical works confirm this scenario [67–69].

According to the standard classification, the critical behavior of the order-disorder transition of diatomic molecules on honeycomb lattice in the p(2×1) structure is described by the $N = 3$ cubic model [5,46]. The experimental investigation [70] of Oxygen on Ru(001) shows that this transition has critical exponents in agreement with the three-state Potts model within about 10%. Several numerical simulations (after the first ones favoring a first order phase transition [71,72]) confirm this critical behavior [73,74]. So it may be possible that a fixed point of the Heisenberg model with cubic anisotropy has the critical behavior of the three-state Potts model.

We want still to note the possibility for systems having exponents close to the four-state Potts model, to belong to the universality class of the XY model with cubic anisotropy and living near the four-state Potts transition characterized by $y = 0$.

III. THE FIXED DIMENSION PERTURBATIVE EXPANSION IN TWO DIMENSIONS

A. Renormalization of the theory

The fixed-dimension field-theoretical approach [16] represents an effective procedure in the study of the critical properties of systems belonging to the $O(N)$ universality class (see, e.g., Ref. [75]). The idea is to extend this procedure to models where there are two ϕ^4 couplings with different symmetry [22,18]. One performs an expansion in powers of appropriately defined zero-momentum quartic couplings and renormalizes the theory by a set of zero-momentum conditions for the (one-particle irreducible) two-point and four-point correlation functions:

$$\Gamma^{(2)}_{ab}(p) = \delta_{ab} Z_\phi^{-1} \left[m^2 + p^2 + O(p^4) \right], \qquad (10)$$

$$\Gamma^{(4)}_{abcd}(0) = Z_\phi^{-2} m^2 \left[\tfrac{u}{3} \left(\delta_{ab}\delta_{cd} + \delta_{ac}\delta_{bd} + \delta_{ad}\delta_{bc} \right) + v\, \delta_{ab}\delta_{ac}\delta_{ad} \right]. \qquad (11)$$

They relate the second-moment mass m, and the zero-momentum quartic couplings u and v to the corresponding Hamiltonian parameters r, u_0, and v_0:

$$u_0 = m^2 u Z_u Z_\phi^{-2}, \qquad v_0 = m^2 v Z_v Z_\phi^{-2}. \qquad (12)$$

In addition, one introduces the function Z_t defined by the relation

$$\Gamma^{(1,2)}_{ab}(0) = \delta_{ab} Z_t^{-1}, \qquad (13)$$

where $\Gamma^{(1,2)}$ is the (one-particle irreducible) two-point function with an insertion of $\frac{1}{2}\phi^2$.

From the pertubative expansion of the correlation functions $\Gamma^{(2)}$, $\Gamma^{(4)}$, and $\Gamma^{(1,2)}$ and the above relations,

one derives the functions $Z_\phi(u,v)$, $Z_u(u,v)$, $Z_v(u,v)$, and $Z_t(u,v)$ as a double expansion in u and v.

The fixed points of the theory are given by the common zeros of the β-functions

$$\beta_u(u,v) = m \left.\frac{\partial u}{\partial m}\right|_{u_0,v_0},$$

$$\beta_v(u,v) = m \left.\frac{\partial v}{\partial m}\right|_{u_0,v_0}. \qquad (14)$$

The stability properties of the fixed points are controlled by the eigenvalues ω_i of the matrix

$$\Omega = \begin{pmatrix} \dfrac{\partial \beta_u(u,v)}{\partial u} & \dfrac{\partial \beta_u(u,v)}{\partial v} \\ \dfrac{\partial \beta_v(u,v)}{\partial u} & \dfrac{\partial \beta_v(u,v)}{\partial v} \end{pmatrix}, \qquad (15)$$

computed at the given fixed point: a fixed point is stable if both eigenvalues are positive. The eigenvalues ω_i are related to the leading scaling corrections, which vanish as $\xi^{-\omega_i} \sim |t|^{\Delta_i}$ where $\Delta_i = \nu\omega_i$.

One also introduces the functions

$$\eta_\phi(u,v) = \left.\frac{\partial \ln Z_\phi}{\partial \ln m}\right|_{u_0,v_0} = \beta_u \frac{\partial \ln Z_\phi}{\partial u} + \beta_v \frac{\partial \ln Z_\phi}{\partial v}, \quad (16)$$

$$\eta_t(u,v) = \left.\frac{\partial \ln Z_t}{\partial \ln m}\right|_{u_0,v_0} = \beta_u \frac{\partial \ln Z_t}{\partial u} + \beta_v \frac{\partial \ln Z_t}{\partial v}. \quad (17)$$

Finally, the critical exponents are obtained from

$$\eta = \eta_\phi(u^*,v^*), \qquad (18)$$

$$\nu = \left[2 - \eta_\phi(u^*,v^*) + \eta_t(u^*,v^*)\right]^{-1}, \qquad (19)$$

$$\gamma = \nu(2-\eta), \qquad (20)$$

where (u^*,v^*) is the position of the stable fixed point.

B. The four loop series

In this section we present the perturbative expansion of the RG functions (14), (16), and (17) up to four loops. The diagrams contributing to the two-point and four-point functions are reported in Ref. [76]. We do not calculate the integrals associated to each diagram, but we use the numerical results compiled in Ref. [76]. Summing all contributions with the right symmetry and group factors (see Ref. [18]) we obtain all RG functions.

The results are written in terms of the rescaled couplings

$$u \equiv \frac{8\pi}{3} R_N \, \bar{u}, \qquad v \equiv \frac{8\pi}{3} \, \bar{v}, \qquad (21)$$

where $R_N = 9/(8+N)$. We adopt this rescaling to have finite fixed point values in the limit $N \to \infty$.

The resulting series are

$$\bar{\beta}_{\bar{u}} = -\bar{u} + \bar{u}^2 + \frac{2}{3}\bar{u}\bar{v} + \bar{u}\sum_{i+j\geq 2} b_{ij}^{(u)} \bar{u}^i \bar{v}^j, \qquad (22)$$

$$\bar{\beta}_{\bar{v}} = -\bar{v} + \bar{v}^2 + \frac{12}{8+N}\bar{u}\bar{v} + \bar{v}\sum_{i+j\geq 2} b_{ij}^{(v)} \bar{u}^i \bar{v}^j, \qquad (23)$$

$$\eta_\phi = \sum_{i+j\geq 2} e_{ij}^{(\phi)} \bar{u}^i \bar{v}^j, \qquad (24)$$

$$\eta_t = -\frac{2(2+N)}{(8+N)}\bar{u} - \frac{2}{3}\bar{v} + \sum_{i+j\geq 2} e_{ij}^{(t)} \bar{u}^i \bar{v}^j \qquad (25)$$

where

$$\bar{\beta}_{\bar{u}} = \frac{3}{16\pi} R_{2N}^{-1} \beta_u, \qquad \bar{\beta}_{\bar{v}} = \frac{3}{16\pi} R_{2N}^{-1} \beta_v. \qquad (26)$$

The coefficients $b_{ij}^{(u)}$, $b_{ij}^{(v)}$, $e_{ij}^{(\phi)}$, and $e_{ij}^{(t)}$ are reported in the Tables I, II, III, and IV. Note that due to the rescaling (26), the matrix element of Ω are two times the derivative of $\bar{\beta}$ with respect to $\bar{u}$ and $\bar{v}$.

We have verified the exactness of our series by the following relations:

(i) $\bar{\beta}_{\bar{u}}(\bar{u},0)$, $\eta_\phi(\bar{u},0)$ and $\eta_t(\bar{u},0)$ reproduce the corresponding functions of the O(N)-symmetric model [77,78];

(ii) $\bar{\beta}_{\bar{v}}(0,\bar{v})$, $\eta_\phi(0,\bar{v})$ and $\eta_t(0,\bar{v})$ reproduce the corresponding functions of the Ising-like ($N=1$) ϕ^4 theory;

(iii) The following relations hold for $N=1$:

$$\bar{\beta}_{\bar{u}}(u,x-u) + \bar{\beta}_{\bar{v}}(u,x-u) = \bar{\beta}_{\bar{v}}(0,x), \qquad (27)$$
$$\eta_\phi(u,x-u) = \eta_\phi(0,x),$$
$$\eta_t(u,x-u) = \eta_t(0,x).$$

(iv) For $N=2$, using the symmetry (3) and (4), and taking into account the rescalings (21), one can easily obtain the identities

$$\bar{\beta}_{\bar{u}}(\bar{u}+\tfrac{5}{3}\bar{v},-\bar{v}) + \frac{5}{3}\bar{\beta}_{\bar{v}}(\bar{u}+\tfrac{5}{3}\bar{v},-\bar{v}) = \bar{\beta}_{\bar{u}}(\bar{u},\bar{v}), \qquad (28)$$
$$\bar{\beta}_{\bar{v}}(\bar{u}+\tfrac{5}{3}\bar{v},-\bar{v}) = -\bar{\beta}_{\bar{v}}(\bar{u},\bar{v}),$$
$$\eta_\phi(\bar{u}+\tfrac{5}{3}\bar{v},-\bar{v}) = \eta_\phi(\bar{u},\bar{v}),$$
$$\eta_t(\bar{u}+\tfrac{5}{3}\bar{v},-\bar{v}) = \eta_t(\bar{u},\bar{v}).$$

These relations are exactly satisfied by our four-loop series. Note that, since the Ising fixed point is $(0,g_I^*)$, and g_I^* is known with very high precision from Ref. [79]

$$g_I^* = 1.7543637(25), \qquad (29)$$

the above symmetry gives us the location of the cubic fixed point: $(\frac{5}{3}g_I^*,-g_I^*)$.

(v) In the large-N limit the critical exponents of the cubic fixed point are related to those of the Ising model: $\eta = \eta_I$ and $\nu = \nu_I$. One can easily see that, for $N \to \infty$,

TABLE I. The coefficients $b_{ij}^{(u)}$, cf. Eq. (22).

i,j	$(N+8)^i b_{ij}^{(u)}$
2,0	$-(10.3350 + 47.675N)$
1,1	-8.39029
0,2	-0.21608
3,0	$524.377 + 149.152N + 5.00028N^2$
2,1	$144.813 + 7.27755N$
1,2	$10.0109 + 0.0583278N$
0,3	0.231566
4,0	$-(7591.108 + 2611.15N + 179.697N^2 + 0.0888427N^3)$
3,1	$-(2872.09 + 291.255N - 0.126814N^2)$
2,2	$-(330.599 + 5.97086N)$
1,3	$-(16.0559 + 0.0578955N)$
4,0	-0.311695

TABLE II. The coefficients $b_{ij}^{(v)}$, cf. Eq. (23).

i,j	$(N+8)^i b_{ij}^{(v)}$
2,0	$-(92.6834 + 5.83417N)$
1,1	-17.392
0,2	-0.716174
3,0	$1228.63 + 118.504N - 1.83156N^2$
2,1	$358.882 + 2.84758N$
1,2	31.4235
0,3	0.930766
4,0	$-(20723.1 + 2692.0N + 25.4854N^2 + 0.824655N^3)$
3,1	$-(8273.28 + 233.78N - 0.574757N^2)$
2,2	$-(1134.8 + 1.91402N)$
1,3	-68.4022
0,4	-1.58239

TABLE III. The coefficients $e_{ij}^{(\phi)}$, cf. Eq.(24).

i,j	$(N+8)^i e_{ij}^{(\phi)}$
2,0	$0.917086\,(N+2)$
1,1	0.611391
0,2	0.033966
3,0	$-0.054609\,(N+2)(N+8)$
2,1	$-0.054609\,(N+8)$
1,2	-0.054609
0,3	-0.002023
4,0	$(N+2)(26.7676 + 4.24178N - 0.092684N^2)$
3,1	$27.6901 + 5.6557N - 0.123579N^2$
2,2	$5.404243 + 0.132800N$
1,3	0.410151
0,4	0.011393

TABLE IV. The coefficients $e_{ij}^{(t)}$, cf. Eq. (25).

i,j	$(N+8)^i e_{ij}^{(t)}$
2,0	$6.751258\,(N+2)$
1,1	4.500839
0,2	0.250047
3,0	$-(96.7105 + 65.1686N + 8.40668N^2)$
2,1	$-(48.3553 + 8.40668N)$
1,2	$-(6.19023 + 0.116656N)$
0,3	-0.233588
4,0	$1135.05 + 844.5N + 139.656N^2 + 0.583377N^3$
3,1	$756.697 + 184.652N + 0.777836N^2$
2,2	$149.467584 + 7.553459N$
1,3	$11.515397 + 0.115791N$
0,4	0.323089

$\eta_\phi(u,v) = \eta_I(v)$, where $\eta_I(v)$ is the perturbative series that determines the exponent η of the Ising model. Therefore, the first relation is trivially true. On the other hand, the second relation $\nu = \nu_I$ is not identically satisfied by the series, and is verified only at the critical point [18].

(vi) For $N = 0$ the series reproduce the results of Ref. [15].

C. Resummations of the series

The field theoretic perturbative expansion generates asymptotic series that must be resummed to extract the physical information about the critical behavior of the real systems.

Exploiting the property that these series are Borel summable for ϕ^4 theories in two and three dimensions [80], one can resum these perturbative expressions considering the Borel transform combined with a method for its analytical extension. In the case of the $O(N)$ symmetric model with only a coupling g, all perturbative series are of the form $F(g) = \sum f_k g^k$. Exploiting the knowledge of the large order behavior of the coefficients f_k (Ref. [75])

$$f_k \sim k!\,(-a)^k\, k^b\, \left[1 + O(k^{-1})\right] \qquad \text{with} \qquad a = -1/g_b,$$
(30)

(a large order behavior related to the singularity g_b of the Borel transform closest to the origin) one can perform the following mapping [81]

$$y(g) = \frac{\sqrt{1 - g/g_b} - 1}{\sqrt{1 - g/g_b} + 1}$$
(31)

to extend the Borel transform of $F(g)$ to all positive values of g. The singularity g_b depends only on the considered model and can be obtained from a steepest-descent calculation in which the relevant saddle point is a finite-energy solution (instanton) of the classical field equations with negative coupling [82,83]. Instead the coefficient b depends on which Green's function is considered.

Note that the function $F(g)$ can be Borel summable only if there are no singularities of the Borel transform on the positive real axis.

This resummation procedure has worked successfully for the $O(N)$ symmetric theory, for which accurate estimates for the critical exponents and other physical quantities have been obtained [81,84,85].

For this reason we want to extend the resummation procedure cited above to multicoupling models, as it has been done for the three-dimensional cubic model [18,78] and for the frustrated system with non-collinear order in two dimensions [86]. Considering a double expansion in $\bar{u}$ and $\bar{v}$ at fixed $z = \bar{v}/\bar{u}$, and studying the large order behavior (following the same procedure used in Refs. [18,86]) of the new expansion in powers of $\bar{u}$ to calculate the singularity of the Borel transform closest to the origin $\bar{u}_b$ we have

$$\frac{1}{\overline{u}_b} = -a\left(R_N + z\right) \qquad \text{for} \qquad 0 < z, \qquad (32)$$

$$\frac{1}{\overline{u}_b} = -a\left(R_N + \frac{1}{N}z\right) \qquad \text{for} \qquad 0 > z > -\frac{2NR_N}{N+1},$$

where $a = 0.238659217\ldots$.

Note that the series in powers of $\overline{u}$ keeping z fixed is not Borel summable for $\overline{u} > 0$ and $z < -R_N$. This fact will not be a real limitation for us, since we will only consider values of z such that $\overline{u}_b < 0$. It should be noted that these results do not apply to the case $N = 0$. Indeed, in this case, additional singularities in the Borel transform are expected [87].

The exponent b in Eq. (30) is related to the number of symmetries broken by the classical solution [83]. It depends on the quantity considered. In the cubic model, for $v \neq 0$, we have $b = 2$ for the function η_ϕ, and $b = 3$ for the β-functions and η_t. For $v = 0$, we recover the results of the O(N)-symmetric model, that is $b = (3 + N)/2$ for η_ϕ, and $b = (5 + N)/2$ for the β-function and η_t [81].

Good estimates of the critical exponent could be obtained also using a Padé Borel analysis of the series, as shown in the case of the two-dimensional O(N) models [77,78] and the random Ising [15].

An important issue in the fixed dimension approach to critical phenomena (and in general of all the field theoretical methods) concerns the analytic properties of the β-functions. As shown in Ref. [89] for the O(N) model, the presence of confluent singularities in the zero of the perturbative β function causes a slow convergence of the resummation of the perturbative series to the correct fixed point value. The O(N) two-dimensional field-theory estimates of physical quantities [81,78] are less accurate than the three-dimensional ones, due to the stronger nonanalyticities at the fixed point [89–92]. In Ref. [89] it is shown that the nonanalytic terms may cause large imprecisions in the estimate of the exponent related to the leading correction to the scaling ω; instead the result for the fixed point value is a rather good approximation of the correct one (if one compare the field theoretical results for the four point renormalized coupling in the g-expansion [78] and in the ϵ-expansion [92,93] with the estimates of other non-perturbative methods [79,94,95], one finds that the systematic error is always less then 10%). We think that this scenario holds also for the cubic models.

IV. FOUR-LOOP EXPANSION ANALYSIS

A. The analysis method

In order to study the critical properties of the continuous anisotropic cubic model (1), we use two different resummation procedures: the Padé-Borel method and the conformal mapping of the Borel-transformed series.

Explicitly, let us consider an l-loop series in $\overline{u}$ and $\overline{v}$ of the form

$$R(\overline{u}, \overline{v}) = \sum_{k=0}^{l} \sum_{h=0}^{l-k} R_{hk} \overline{u}^h \overline{v}^k = \sum_{k=0}^{l} R_k(z) \overline{u}^k, \qquad (33)$$

where R is one of the RG functions and z is the ratio $\overline{v}/\overline{u}$ that we will consider always fixed to same value. In this manner we have an asymptotic series of only one variable $\overline{u}$, depending on the additional parameter z that have to run from 0 to ∞ in order to reproduce all the quadrant with $\overline{u}, \overline{v} \geq 0$. In order to use a variable defined in a finite range we set

$$z = \tan\frac{\pi}{2}x, \qquad (34)$$

with $x \in [0, 1]$. In the follow we fix $x = k/N_{max}$, where k is an integer number running from 0 to N_{max} (we will fix $N_{max} = 25$).

In the Padé-Borel method we consider the Padé approximants $[L/M]$ of the Borel-Leroy transform of $R(\overline{u}, z)$. Explicitly the approximants of R are

$$E(R)(L, M, b; \overline{u}, z) = \int_0^\infty dt\, t^b e^{-t} \frac{N_L(\overline{u}t, z)}{D_M(\overline{u}t, z)} \qquad (35)$$

where N_L and D_M are two polynomials of degree L and M respectively, with $L + M \leq l$, that are determined by the condition that the expansion of $E(R)(L, M, b; \overline{u}, z)$ in powers of $\overline{u}$ gives $R(\overline{u}, z)$ to order $L + M$.

In this manner we have several approximants of the function $R(\overline{u}, \overline{v})$ with varying the three parameters b, L, and M. As usual in this case [78] the best estimates of the resummed function are given by the diagonal and near-diagonal approximants (i.e. for our four-loop series we have three reasonable choices $[2, 2]$, $[3, 1]$, and $[2, 1]$). Then we search the value of b (called b_{opt}) integer or half-integer minimizing the differences between the considered approximants; a reasonable estimate will be the mean value of $E(b_{opt}, L, M)$ on all the values of L and M considered. Then we could take as error bar the deviations from the mean value of all the approximants with $b_{opt} - 1 \leq b \leq b_{opt} + 1$ (always considering integer and half-integer values of b).

In the analysis using the conformal-mapping method we essentially follow the procedure used in Ref. [18]. We exploit the knowledge of the value of the singularity of the Borel transform closest to the origin (a value given in the previous section), and we generate a set of approximants to our asymptotic series, varying the two parameters α and b appearing in

$$E(R)_p(\alpha, b; \overline{u}, z) = \sum_{k=0}^{l} B_k(\alpha, b; z)$$

$$\times \int_0^\infty dt\, t^b e^{-t} \frac{y(\overline{u}t; z)^k}{[1 - y(\overline{u}t; z)]^\alpha}, \qquad (36)$$

TABLE V. Half of the exponent ω_2 at the $O(N)$ fixed point. CM is the value obtained using conformal mapping technique and PB the one using Padé-Borel.

N	$u^*_{O(N)}$	CM	PB
2	1.80(3)	0.03(3)	0.06(4)
3	1.75(2)	−0.08(3)	−0.07(3)
4	1.70(2)	−0.18(4)	−0.17(5)
8	1.52(1)	−0.45(5)	−0.44(6)

where

$$y(x;z) = \frac{\sqrt{1 - x/\overline{u}_b(z)} - 1}{\sqrt{1 - x/\overline{u}_b(z)} + 1}. \qquad (37)$$

The coefficients B_k are determined by the condition that the expansion of $E(R)_p(\alpha, b; \overline{u}, z)$ in powers of $\overline{u}$ and $\overline{v}$ gives $R(\overline{u}, \overline{v})$ to order l.

The procedure to choose the range of the parameters α and b used to find good estimates and reasonable error bars of the various quantities is the same of Ref. [18].

B. Stability properties of the $O(N)$ and the Ising fixed points

First of all, we analyze the stability properties of the $O(N)$-symmetric fixed point. Since

$$\frac{\partial \beta_{\overline{v}}}{\partial \overline{u}}(\overline{u}, 0) = 0, \qquad (38)$$

the eigenvalues are simply

$$\omega_1 = 2\frac{\partial \bar{\beta}_{\overline{u}}}{\partial \overline{u}}(\overline{u}^*, 0), \qquad \omega_2 = 2\frac{\partial \bar{\beta}_{\overline{v}}}{\partial \overline{v}}(\overline{u}^*, 0), \qquad (39)$$

where $\overline{u}^*$ is the fixed-point value of the $O(N)$ vector model of which accurate estimates are available from the five-loop analysis of Ref. [78]. The exponent ω_1 is the usual exponent of the $O(N)$-symmetric theory [75] that gives the first correction to the scaling. For all $N > 2$ it is known that ω_1 assumes the constant value 2 (see Ref. [89] and references therein). Instead ω_2 is the eigenvalue that determines the stability of the fixed point respect to an anisotropic cubic perturbation.

In Table V we report the results for ω_2 for several values of N. It is quite evident that the $O(N)$ fixed point is unstable for $N \geq 3$ (we note that for small N these values are very close to zero). In the limit $N \to \infty$ it holds $\omega_2 = -2$. For $N = 2$ our result is compatible with the presence of a marginal operator, i.e., $\omega_2 = 0$.

Then we focus our attention on the stability properties of the Ising fixed point. Also in this case the eigenvalues are simply:

$$\omega_1 = 2\frac{\partial \bar{\beta}_{\overline{v}}}{\partial \overline{v}}(0, \overline{v}^*), \qquad \omega_2 = 2\frac{\partial \bar{\beta}_{\overline{u}}}{\partial \overline{u}}(0, \overline{v}^*), \qquad (40)$$
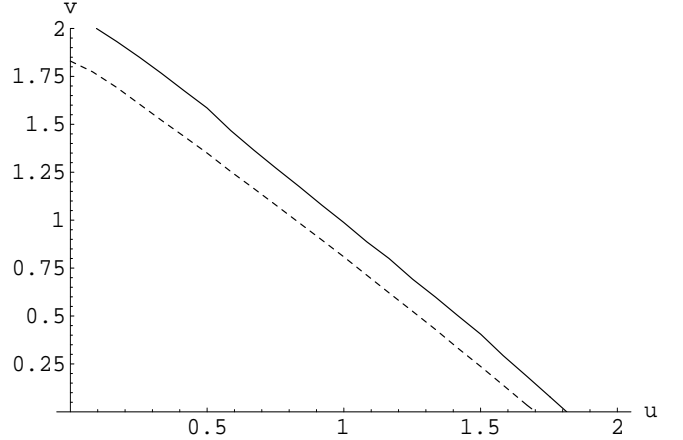

FIG. 2. Zeros of the $\bar{\beta}$-functions for $N = 2$ in the $(\overline{u}, \overline{v})$ plane using the conformal mapping method. The straight line correspond to zeros of $\bar{\beta}_{\overline{u}}(\overline{u}, \overline{v})$ and the dashed one to the zeros of $\bar{\beta}_{\overline{v}}(\overline{u}, \overline{v})$.

where $\overline{v}^*$ is the fixed-point value of the Ising model (29). The exponent ω_1 gives the correction to the scaling of the Ising model and it should be equal to 7/4 [89], while ω_2 is the eigenvalue determining the stability of this fixed point. We find that the series $\omega_2(\overline{v})$ is independent from N (we already know that its fixed point value must be equal to α_I for all N):

$$\frac{\omega_2^I(\overline{v})}{2} = -1 + \frac{2}{3}\overline{v} - 0.2161\overline{v}^2 + 0.23157\overline{v}^3 - 0.31169\overline{v}^4. \qquad (41)$$

The fixed point value of this exponent is $\omega_2^I/2 = -0.10(5)$, using the conformal mapping method, and $-0.08(5)$, using the Padé-Borel analysis. These values are quite close to $-\alpha_I = 0$; we attribute this small discrepancy to the presence of nonanalyticities at the zero of the $\bar{\beta}_{\overline{v}}$. To support this thesis we note that the value of $\omega_2/2$ is approximately the same of the resummation of the exponent $-\alpha_I = -0.081$ found in Ref. [78].

C. Evaluation of non-trivial fixed points

Since the Ising fixed point has always a marginal operator and the $O(N)$ is unstable for $N \geq 3$, we search for the presence of non-trivial fixed points in the region $\overline{u}, \overline{v} > 0$.

In Figs. 2 and 3 we show the zeros of the β functions for N=2 using the conformal mapping technique and the Padé-Borel analysis respectively. The curves of zeros of $\bar{\beta}_{\overline{u}}$ and $\bar{\beta}_{\overline{v}}$ seem parallel and do not have any intersection (in these figures, and in all the others, the error bar are not shown to make them more readable). The distance between the two lines is always less than the error bar (that is of the order of 0.1 for each line). So our result is compatible with the presence of the line of fixed

points conjectured by Kadanoff [41], joining the decoupled Ising and the $O(2)$ fixed points. In the Kadanoff identification this line of fixed points is the dual line of the 8V model in the region that goes from the F-model limit ($O(2)$ multicritical point with $y = 3/2$) to the decoupled Ising point ($y = 1$). This line of fixed points is identifiable with the one of Ref. [35] with $h_4 < 0$.

For the symmetry (4) the fixed-point line continues for negative $\overline{v}$ and reaches the "cubic" fixed point (that in this case is a standard Ising) located at $(5/3\,g_I^*, -g_I^*)$. This second line is the one of Ref. [35] with $h_4 > 0$ that is isomorphic to the other one with $h_4 < 0$. We do not know if the line continues in the region with negative $\overline{u}$, that may be a region of second order phase transition, having the critical exponent of the 8V model with $y < 1$ (only for $v > -u$, from the stability condition). We tried to perform the resummation also for these value of $\overline{u}, \overline{v}$ but we obtained strongly oscillating results.

For the other values of N we find one fixed point at the intersection of the two curves of zeros of the β-functions, that is the usual cubic fixed point, analytic continuation of the one found in the ϵ-expansion. For example the results of the conformal mapping analysis for $N = 3$ and of the Padé-Borel for $N = 4$ are shown in Figs. 4 and 5 respectively. All our results about the cubic fixed points are summarize in Table VI. We note that, within the error bar, there is full agreement between the two methods [96].

In the limit $N \to \infty$ the series $\bar{\beta}_{\overline{u}}(\overline{u}, \overline{v})$ and $\bar{\beta}_{\overline{v}}(\overline{u}, \overline{v})$ simplify to

$$\bar{\beta}_{\overline{v}}(\overline{u}, \overline{v}) \to \quad \bar{\beta}_{\text{Ising}}(\overline{v}), \tag{42}$$

$$\bar{\beta}_{\overline{u}}(\overline{u}, \overline{v}) \to \overline{u}[P_1(\overline{v}) - \overline{u}P_2(\overline{v})], \tag{43}$$
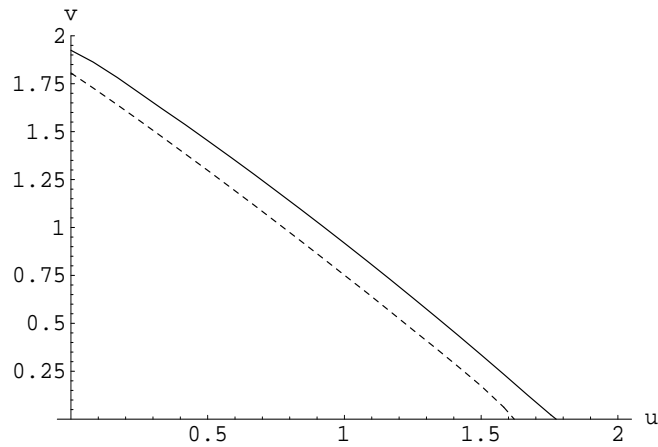
with

FIG. 3. Zeros of the $\bar{\beta}$-functions for $N = 2$ in the $(\bar{u}, \bar{v})$ plane using the Pade-Borel analysis. The straight line correspond to zeros of $\bar{\beta}_{\bar{u}}(\bar{u}, \bar{v})$ and the dashed one to the zeros of $\bar{\beta}_{\bar{v}}(\bar{u}, \bar{v})$.
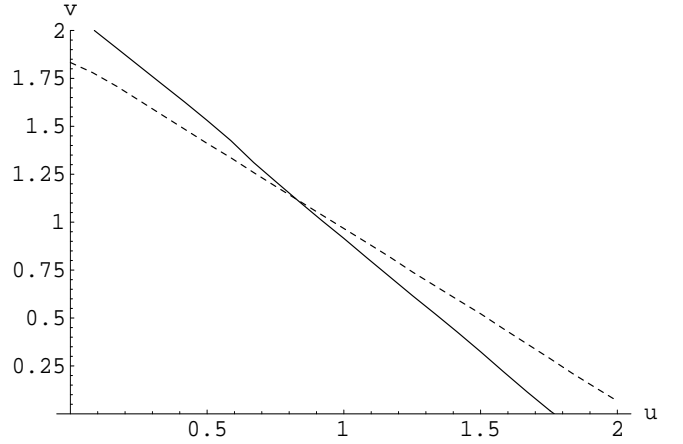

FIG. 4. Zeros of the $\bar{\beta}$-functions for $N = 3$ in the $(\bar{u}, \bar{v})$ plane using the conformal mapping method. The straight line correspond to zeros of $\bar{\beta}_{\bar{u}}(\bar{u}, \bar{v})$ and the dashed one to the zeros of $\bar{\beta}_{\bar{v}}(\bar{u}, \bar{v})$.

$$P_1(\overline{v}) = -1 + \frac{2}{3}\overline{v} - 0.21608\overline{v}^2 + 0.23157\overline{v}^3 - 0.31168\overline{v}^4 \tag{44}$$

and

$$P_2(\overline{v}) = 1 + 0.058328\overline{v}^2 - 0.057896\overline{v}^3. \tag{45}$$

The zero of $\bar{\beta}_{\overline{v}}(\overline{u}, \overline{v})$ is constant with varying $\overline{u}$ at the value $\overline{v}^* = \bar{g}_I^*$ (29), and so the $\overline{u}$ coordinate of the cubic fixed point is simply:

$$u_{N\to\infty}^* = \frac{P_1(\bar{g}_I^*)}{P_2(\bar{g}_I^*)} = 0.09(4). \tag{46}$$

We note that the small value of u^* for $N \to \infty$ does not exclude that the Ising and the cubic fixed points coalesce in this limit.
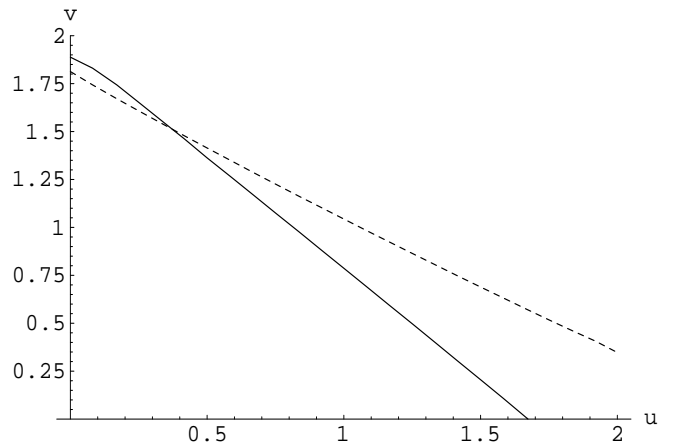

FIG. 5. Zeros of the $\bar{\beta}$-functions for $N = 4$ in the $(\bar{u}, \bar{v})$ plane using the Pade-Borel analysis. The straight line correspond to zeros of $\bar{\beta}_{\overline{u}}(\overline{u}, \overline{v})$ and the dashed one to the zeros of $\bar{\beta}_{\overline{v}}(\overline{u}, \overline{v})$.

TABLE VI. Cubic fixed points for $N \geq 3$.

N	CM $(\bar{u}^*, \bar{v}^*)$	PB $(\bar{u}^*, \bar{v}^*)$
8	(0.24(8),1.72(10))	(0.15(9),1.72(12))
4	(0.54(10),1.43(8))	(0.38(11),1.51(9))
3	(0.83(12),1.12(9))	(0.71(12),1.17(9))

TABLE VII. The half of eigenvalues of the Ω matrix for $N \geq 3$ at the cubic fixed point.

N	CM $(\omega_1/2, \omega_2/2)$	PB $(\omega_1/2, \omega_2/2)$	pseudo-ϵ $(\omega_1/2, \omega_2/2)$
∞		(0.64(10), 0.11(8))	(0.65(3), 0.07(4))
8		(0.70(10), 0.07(6))	(0.72(3), 0.08(4))
4	(0.65(15), 0.11(6))	(0.70(15), 0.06(6))	(0.79(4), 0.06(4))
3	(0.70(7), 0.09(6))	(0.70(10), 0.04(5))	(0.75(6), 0.05(5))

We want to add that in Ref. [54], using an argument based on conformal field theory, it is conjectured that the cubic fixed point and the Ising one merge for arbitrary N. Our analysis seems to contradict this statement.

D. Stability properties of the cubic fixed point

For $N = 2$ we check that one eigenvalue of the stability matrix Ω vanish on all the line of fixed points. From Figs. 2, 3 it is evident that this line is almost straight, so for our calculation we evaluate the Ω matrix on the line joining the Ising fixed point (at $(0, g_I^*)$) and the $O(2)$ (at $(1.80(3), 0)$ from Ref. [78]). Within the precision of our calculation we confirm that on this line one marginal operator exists (see Fig. 6, the uncertainty of $\omega_2/2$ is constantly of the order of 0.05).

The results for the eigenvalues of the Ω matrix at the cubic fixed point for $N \geq 3$ are summarize in Table VII. The eigenvalue ω_2 is positive for all considered values of N, but it is very close to zero. Being the Ising fixed point stable against perturbations with positive u, it is not possible that the cubic fixed point is stable for "left" perturbations. We believe that our results are compatible with the presence of a marginal operator also at the cubic fixed point and that the very small discrepancy of ω_2 with respect to zero is due to nonanalytic terms.

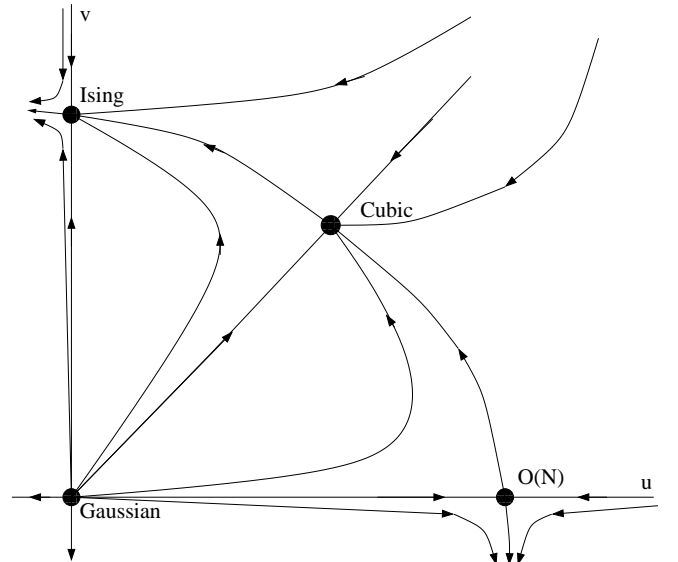We conjecture that the RG flow in the coupling plane u, v for $N \geq 3$ is the one sketched in Fig. 7. For $u, v < 0$ we expect a first order phase transition. For $0 < v/u \leq v^*/u^*$ the transition is second order and its critical behavior is characterized by the cubic fixed point. For $v/u \geq v^*/u^*$ the transition is governed by the Ising fixed point, i.e., with the exponents of the Ising model more logarithmic corrections.

To support this scenario we also consider the pseudo-ϵ expansion using substantially the same procedure of Ref. [27]. By using this trick we have a smallest error bar since we avoid the uncertainty of the fixed point (for a detailed discussion see Ref. [81]). The results are equivalent to the ones presented above (see Table VII).

The presence of a marginal operator both at Ising and at cubic fixed points means that we should expect slowly decaying crossover effects that could produce systematic errors in Monte Carlo simulations and in the experimental investigations.

E. Critical exponents

The direct evaluation of the critical exponents for the two-dimensional $O(N)$ model leads to erroneous values because of the strong effect of nonanalytic terms. In fact

FIG. 6. Values of $\omega_1/2$ (dashed line) and $\omega_2/2$ (straight line) with varying the parameter x (34).

FIG. 7. Conjectured renormalization-group flow in the coupling plane (u, v) for $N > 2$.

from the analysis of the four- and five-loop series of the $N = 1$ model is found $\eta \sim 0.131$ [78] and $\eta = 0.13(7)$ [81] instead of the exactly known $\eta = 1/4$. The value of ν for the Ising universality class is quite good ($\nu \sim 0.96$ [81,78]). An our unpublished analysis of the $O(N)$ series (already cited in Ref. [86]) shows that for higher values of N the effect of nonanalyticities is very dangerous. In fact, studying four-loop series, we find $\eta = 0.11(6)$ both for the XY and Heisenberg model instead of $1/4$ and 0. In the same way for $\eta - \eta_t$ we find 1.18(5) and 1.36(4), for $N = 2, 3$, instead of 2. This latter systematic error brings to a finite value of ν. The above standard analysis applied to the XY model with cubic anisotropy gives $\eta \sim 0.11$ along the fixed-point line, and values ranging from 1.18 to 0.97 for $\eta - \eta_t$.

In order to reduce the effect of nonanalytic terms in the estimates of the critical exponents of the cubic model we adopt a new strategy. We use a constrained analysis on the two variables series η and $\eta - \eta_t$ fixing the values assumed at the fixed point on the axes $\overline{u} = 0$ and $\overline{v} = 0$. Explicitly, let us consider a generic l-loop series in $\overline{u}$ and $\overline{v}$ without constant term

$$R(\bar{u}, \bar{v}) = \sum_{k=0}^{l} \sum_{h=0}^{l-k} R_{hk} \overline{u}^h \overline{v}^k = \sum_{k=1}^{l} R_k(z) \overline{u}^k, \qquad (47)$$

of which we know the values $R(\overline{u}^*, 0) = a$ and $R(0, \overline{v}^*) = b$. We can rewrite the previous function in the form

$$R(\overline{u}, \overline{v}) = R(\overline{u}, 0) + R(0, \overline{v}) + \overline{u}\,\overline{v} A_R(\overline{u}, \overline{v}) \qquad (48)$$

where $\overline{u}\,\overline{v} A_R(\overline{u}, \overline{v})$ is the difference between the original function $R(\overline{u}, \overline{v})$ and the value that it assumes on the axes. We resum the three functions $R(\overline{u}, 0)$, $R(0, \overline{v})$, and $A_R(\overline{u}, \overline{v})$, appearing in (48), in an independent way. In the case of $R(\overline{u}, 0)$ and $R(0, \overline{v})$ we adopt the standard method to resum one variable functions with some constraints, as done for the ϵ-expansion in Refs. [84,92,18]. A full description of this method may be found in Ref. [92].

First of all we consider the peculiar case $N = 2$. We evaluate the critical exponents on the straight line joining the Ising and the $O(2)$ fixed points (as in the case of the Ω matrix). The results for the exponents η and $\eta - \eta_t$ are presented in the Figs. 8 and 9 respectively. Within the precision of our calculation we find an exponent η assuming the constant value $1/4$ along the line as predicted in Ref. [35] (the uncertainty is about 0.05 for $x \sim 0.5$ and decreases near the borders). The exponent $\eta - \eta_t$ interpolates in a monotonic and continuous way from 1 at the Ising fixed point to 2 at the $O(2)$ fixed point and so it is compatible with the Kadanoff's conjecture identifying this line with the dual line of the 8V model with the continuous varying exponent given by (7).

The parameter y in equation (7) belongs to the range $[1, 3/2]$ respectively from Ising to $O(2)$, instead x ranges between 0 and 1 from $O(2)$ to Ising. So we try to identify
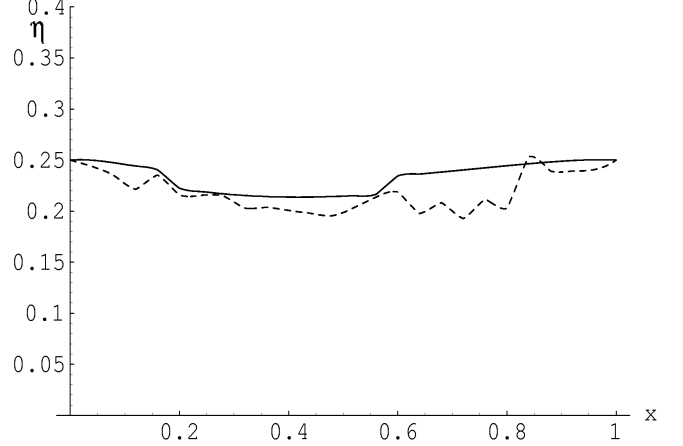

FIG. 8. Values of η with varying the parameter x (34). The straight line represents CM results and the dashed the PB ones.

$$y = \frac{3 - x}{2}. \qquad (49)$$

Using this assumption the exponent $\eta - \eta_t$ could be written in terms of x as

$$\eta - \eta_t = \frac{2}{1 + x}. \qquad (50)$$

This curve is shown in Fig. 9 together with the numerical values obtained from the resummation of the perturbative series: all the curves are very close. This result strongly supports both Kadanoff's conjecture about the nature of the line of fixed points and the relation (49) between the parameter y of the 8V-model and x of the XY model with cubic anisotropy.

The exponents at the cubic fixed point obtained using this constrained analysis for some $N \geq 3$ are reported in Table VIII. We find an η exponent varying very slowly with N. Our estimates are also compatible with an η constant and equal to $1/4$ for all N. The estimates of
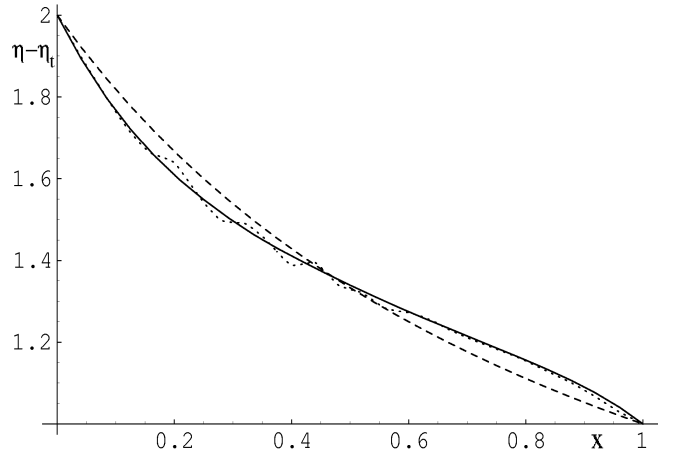

FIG. 9. Values of $\eta - \eta_t$ (straight line CM and pointed line PB) with varying the parameter x (34). The dashed line is our conjecture (50).

11

$\eta - \eta_t$ are instead affected by a big error bar, mainly due to the uncertainty of the cubic fixed points. We could reduce the last error using the pseudo-ϵ expansion, but in this way we will find again the systematic error of nonanalyticities that we cannot control.

From Table VIII it is clear that when N increases the exponents get closer to the ones of the Ising model, that is the limit for $N \to \infty$, cf. (2). In this limit the series η_ϕ reproduces order by order the one of the Ising model. For the difference $\eta - \eta_t$ the equality of the exponent is expected only at the fixed point. In fact it holds

$$\lim_{N \to \infty} \eta_t(\overline{u}, \overline{v}) = -2\overline{u} + \eta_t^I(\overline{v}) - \overline{u}\,\overline{v}^2 A(\overline{v}) \qquad (51)$$

where $\eta_t^I(\overline{v})$ is the series of the Ising model, and $A(\overline{v})$ a function that at four-loop reads $A(\overline{v}) = 0.1166 - 0.1157\,\overline{v}$. At the cubic fixed point we obtain

$$\eta_t - \eta_t^I = 0.13(10) \qquad (52)$$

that is compatible with zero.

For $N = 3$ the result for $\eta - \eta_t$ via scaling law leads to $\nu = 1.3(3)$, so there are no fixed points (stable or unstable) with the three-state Potts exponent $\nu = 5/6$. In Sec. II C we have stressed that experimental investigations [70] and Monte Carlo simulations [73,74] in the adsorption of diatomic molecules on honeycomb lattice in the $p(2 \times 1)$ structure display the critical behavior of three-state Potts, although, according to the standard classification [5], this system should be described by the $N = 3$ cubic model. It may be possible that the system undergoes a weak first-order phase transition in the face-cubic region with $v < 0$, with effective exponents close to the ones of the three-state Potts model which appears in the discrete face-centered model. This point needs further studies for a full understanding of the phenomenon.

V. CONCLUSIONS

In the present paper we have studied the critical behavior of N-component spin models with cubic anisotropy by applying the field-theoretic renormalization-group technique directly in two-dimensions.

We have firstly focused our attention to the XY model ($N = 2$) with cubic anisotropy. We found that this model has a line of fixed points joining the decoupled Ising and the $O(2)$. Along this line the critical exponent

η assumes the constant value $1/4$. The exponent ν runs from 1, at Ising, to ∞, at $O(2)$, according to the standard expression of the 8V model (7). This argument is based on a mapping of the y parameter of the 8V model onto the x parameter of the Hamiltonian (1):

$$x = 3 - 2y = \frac{2}{\pi} \arctan \frac{\overline{v}^*}{\overline{u}^*}. \qquad (53)$$

The predicted expression for ν (Eq. (50)) is in very good agreement with the resummation of perturbative series (see Fig. 9). In this manner we relate the measured exponents to the strength of the anisotropy (the parameter x) [97]. The XY model with cubic anisotropy for $u < 0$ could display the critical behavior of the Askin-Teller model with $y < 1$, i.e., the region on the dual line from the decoupled Ising fixed point ($y = 1$) to the four-state Potts ($y = 0$). We tried to check if the line of fixed points continues for negative value of u, but the results of the resummation for these values of the renormalized couplings are strongly oscillating.

For all other values of N we found one fixed point in the region with $\overline{u}, \overline{v} > 0$, that is the usual cubic fixed point, analytic continuation of one found in the ϵ-expansion. The fixed point is marginally stable and unstable for perturbations with $v/u < v^*/u^*$ and $v/u > v^*/u^*$ respectively. The conjectured RG flow diagram is sketched in Figure 7. The estimates of this fixed point for several values of N are reported in Table VI and the critical behavior is characterized by the exponents of Table VIII.

In particular for the physical relevant case of $N = 3$ at the cubic fixed point we found

$$\eta = 0.17(8), \qquad \nu = 1.3(3). \qquad (54)$$

The value found for ν is different from the one of the three-state Potts model. This fact deserves further studies about the order-disorder transitions of some adsorbed monolayers that should belong to the universality class of the Heisenberg cubic model, but that experimentally display the exponents of the three-state Potts model. One possible scenario is that the system undergoes a very weak first-order phase transition with effective exponents close to the ones of three-state Potts model.

TABLE VIII. Critical exponents for $N \geq 3$ at the cubic fixed point.

N	η [CM,PB]	$\eta - \eta_t$ [CM,PB]
8	[0.26(2),0.25(2)]	[1.20(15),1.15(15)]
4	[0.23(4),0,20(6)]	[1.22(15),1.18(20)]
3	[0.17(8),0.15(9)]	[1.39(20),1.23(20)]

[1] A. Aharony, in *Phase Transitions and Critical Phenomena*, edited by C. Domb and J. Lebowitz (Academic Press, New York, 1976), Vol. 6, p. 357.

[2] A. Pelissetto and E. Vicari, "Critical phenomena and Renormalization Group theory", e-print cond-mat/0012164

[3] In this work we do not consider the quadratic term with cubic symmetry, that may exist for $N = 2$, $\sum_{\mu=1}^{2}(\partial_\mu\phi_\mu)^2$ [98–101,1]. As we shall see, the quartic term in the Hamiltonian (1) already introduces significant changes in the critical behavior.

[4] E. Domany, M. Schick, and J. S. Walker, Phys. Rev. Lett. **38**, 1148 (1977); E. Domany, M. Schick, J. S. Walker, and R. B. Griffiths, Phys. Rev. **B 18**, 2209 (1978).

[5] E. Domany and M. Schick, Phys. Rev. **B 20**, 3828 (1979).

[6] C. Rottman , Phys. Rev. **B 24**, 1482 (1981); ibid. **B 26**, 5141 (1982); J. Bruno and M. R. Giri, ibid. **B 26**, 6711 (1982).

[7] N. C. Bartelt, T. L. Einstein, and L. T. Wille, Phys. Rev. **B 40**, 10759 (1989).

[8] T. Aukrust, M. A. Novotny, P. A. Rikvold, and D. P. Landau, Phys. Rev. **B 41**, 8772 (1990).

[9] T. Chou and D. R. Nelson, Phys. Rev. **E 48**, 4611 (1993).

[10] A. Szolnoki, G. Szabo, and O. G. Mouritsen, Phys. Rev. **E 55**, 2255 (1997).

[11] S. Bekhechi, A. Benyoussef, and N. Moussa, Phys. Rev. **B 61**, 3362 (2000).

[12] A. B. Harris and T. C. Lubensky, Phys. Rev. Lett. **33**, 1540 (1974).

[13] T. C. Lubensky, Phys. Rev. **B 11**, 3573 (1975).

[14] D. E. Khmelnitskii, Zh. Eksp. Teor. Fiz. **68**, 1960 (1975) [Sov. Phys. JETP **41**, 981 (1976)].

[15] I. O. Mayer, J. Phys. **A 22**, 2815 (1989).

[16] G. Parisi, Cargèse Lectures (1973) (unpublished); J. Stat. Phys. **23**, 49 (1980).

[17] A. Aharony, Phys. Rev. **B 8**, 4270 (1973)

[18] J.M. Carmona, A. Pelissetto, and E. Vicari, Phys. Rev. B **61**, 15136 (2000).

[19] J. Sak, Phys. Rev. **B 10**, 3957 (1974).

[20] J. Rudnick, Phys. Rev. **B 11**, 3397 (1975); **B 18**, 1406 (1978); P. Arnold, S. Sharpe, L. G. Yaffe, and Y. Zhang, Phys. Rev. Lett. **78**, 2062 (1997); N. Tetradis, Phys. Lett. **B 431**, 380 (1998).

[21] N. A. Shpot, Phys. Lett. **A 142**, 474 (1989).

[22] I. O. Mayer, A. I. Sokolov, and B. N. Shalayev, Ferroelectrics **95**, 93 (1989).

[23] H. Kleinert and V. Schulte-Frohlinde, Phys. Lett. **B 342**, 284 (1995); H. Kleinert and S. Thoms, Phys. Rev. **D 52**, 5926 (1995); H. Kleinert, S. Thoms, and V. Schulte-Frohlinde, Phys. Rev. **B 56**, 14428 (1997).

[24] B. N. Shalaev, S. A. Antonenko, and A. I. Sokolov, Phys. Lett. **A 230**, 105 (1997).

[25] A. I. Mudrov and K. B. Varnashev, Phys. Rev. **E 58**, 5371 (1998); K. B. Varnashev, Phys. Rev. **B 61**, 14660 (2000).

[26] D. V. Pakhnin and A. I. Sokolov, Phys. Rev. B **61** 15130 (2000).

[27] R. Folk, Yu. Holovatch, and T. Yavors'kii, Phys. Rev. **B 62**, 12195 (2000).

[28] A. Aharony, J. Phys. **A 10**, 389 (1977).

[29] D. J. Wallace, J. Phys. **C 6**, 1390 (1973).

[30] J. Sznajd and M. Dudziński, Phys. Rev. **B 59**, 4176 (1999).

[31] V. J. Emery, Phys. Rev. **B 11**, 239 (1975).

[32] M. E. Fisher, Phys. Rev. **176**, 257 (1968).

[33] A. Aharony, Phys. Rev. Lett. **31**, 1494 (1973).

[34] A. L. Korzhenevskii, Zh. Eksp. Teor. Fiz. **71**, 1434 (1976) [Sov. Phys. JETP **44**, 751 (1976)].

[35] J. V. José, L. P. Kadanoff, S. Kirkpatrick, and D. R. Nelson, Phys. Rev. **B 16**, 1217 (1977);ibid. **B 17**, 1477(E) (1978).

[36] K. E. Newman and E. K. Riedel, Phys. Rev. **B 25**, 264 (1982); E. Domany and E. K. Riedel, J. Appl. Phys. **50**, 1804 (1979).

[37] R. A. Pelcovits and D. R. Nelson, Phys. Lett. **57 A**, 23 (1976).

[38] F. J. Wegner, in *Phase Transitions and Critical Phenomena*, edited by C. Domb and J. Lebowitz (Academic Press, New York, 1976), Vol. 6, p. 8.

[39] J.M. Kosterlitz and D.J. Thouless, J. Phys. **C 6**, 1181 (1973); J.M. Kosterlitz, ibid. **7**, 1046 (1974).

[40] M. Suzuki, Prog. Theor. Phys., **51**, 1992 (1974).

[41] L. P. Kadanoff, Phys. Rev. Lett. **39**, 903 (1977).

[42] R. J. Baxter, *Exactly solved models in statistical mechanics* (Academic Press, San Diego, 1982).

[43] J. Salas and A. D. Sokal, J.Stat.Phys. **85**, 297 (1996).

[44] D. Kim, P. M. Levy, and L. F. Uffer, Phys. Rev. **B 12**, 989 (1975).

[45] D. Kim and P. M. Levy, Phys. Rev. **B 12**, 5105 (1975).

[46] E. Domany and E. K. Riedel, Phys. Rev. Lett. **40**, 561 (1978); Phys. Rev. **B 19**, 5817 (1979).

[47] E. K. Riedel, Physica **106A** , 110 (1981).

[48] B. Nienhuis, E. K. Riedel, and M. Schick, Phys. Rev. **B 27**, 5625 (1983).

[49] G. S. Grest and M. Widom, Phys. Rev. **B 24**, 6508 (1981).

[50] E. Fradkin, Phys. Rev. Lett. **53**, 1967 (1984).

[51] A. J. Bray, Phys. Rev. Lett. **54**, 1593 (1985).

[52] R. Shankar, Phys. Rev. Lett. **55**, 453 (1985).

[53] B. N. Shalaev, Sov. Phys. Solid State **31**, 51 (1989).

[54] B. N. Shalaev, Phys. Rep. **237**, 129 (1994).

[55] G. Jug and B. N. Shalaev, Phys. Rev. **B 54**, 3442 (1996).

[56] P. Ruján, G. O. Williams, H. L. Frisch, and G. Forgács, Phys. Rev. **B 23**, 1362 (1981). In literature the name $Z(p)$ model is often reserved to the p-state clock model or vector Potts model [102]. These systems are distinct from the discrete cubic models for all $p = 2N \neq 4$; they are related to the planar model with p-th order anisotropy of Eq. (5).

[57] Z. Q. Qiu, J. Pearson, and S. D. Bader, Phys. Rev. Lett. **67**, 1646 (1991).

[58] Y. P. Feng and M. H. W. Chan, Phys. Rev. Lett. **71**, 3822 (1993).

[59] M. Michailov, C. de Beauvais, D.Rouxel, and B. M. Mutaftschiev, Phys. Rev. **B 61**, 5987 (2000).

[60] M. Bretz, Phys. Rev. Lett. **38**, 501 (1977).

[61] M. Sokolowski and H. Pfnur, Phys. Rev. **B 49**, 7716

(1994).

[62] P. Piercy and H. Pfnur, Phys. Rev. Lett. **59**, 1124 (1987).

[63] K. Grzelakowski, I. Lyuksyutov, and E. Bauer, Phys. Rev. Lett. **64**, 32 (1990).

[64] D. H. Baek, J. W. Chung, and W. K. Han, Phys. Rev. **B 47**, 8461 (1993).

[65] P. Bak, P. Kleban, W. N. Unlert, J. Ochab, G. Akinci, N. C. Bartlet, and T. L. Einstein, Phys. Rev. Lett. **54**, 1539 (1985).

[66] It should be noted that in Refs. [63,64] is found $\eta = 0.38(12)$ and $\eta \sim 0.57$ respectively. These results are clearly different from the expected $\eta = 1/4$.

[67] K. Kaski, W. Kinzel, and J.D.Gunton, Phys. Rev. **B 27**, 6777 (1983).

[68] P. Piercy, K. De'Bell, and H. Pfnur, Phys. Rev. **B 45**, 1869 (1992).

[69] M. Presber, B. Dunweg, and D.P. Landau, Phys. Rev. **E 58**, 2616 (1998).

[70] P. Piercy and H. Pfnur, Phys. Rev. **B 41**, 582 (1990).

[71] O. G. Mouritsen and A. J. Berlinsky, Phys. Rev. Lett. **48**, 181 (1982);

[72] P. Tarazona and E. Chacón Phys. Rev. **B 39**, 7111 (1989),ibid. **B 39**, 7157 (1989).

[73] Zhi-Xiong Cai, Phys. Rev. **B 43**, 6163 (1991).

[74] L. D. Roelofs and C. Jackson, Phys. Rev. **B 47**, 197 (1993).

[75] J. Zinn-Justin, *Quantum Field Theory and Critical Phenomena*, third edition (Clarendon Press, Oxford, 1996).

[76] B. G. Nickel, D. I. Meiron, and G. A. Baker, Jr., "Compilation of 2-pt and 4-pt graphs for continuum spin models", University Guelph Report, 1977, unpublished.

[77] G. A. Baker, Jr., B. G. Nickel, M. S. Green, and D. I. Meiron, Phys. Rev. Lett. **36**, 1351 (1976); G. A. Baker, Jr., B. G. Nickel, and D. I. Meiron, Phys. Rev. **B 17**, 1365 (1978).

[78] E.V. Orlov and A.I. Sokolov, Fiz. Tverd. Tela **42**, 2087 (2000) (a shorter English version appears as Preprint hep-th/0003140).

[79] M. Caselle, M. Hasenbusch, A. Pelissetto, and E. Vicari, J. Phys. **A 33**, 8171 (2000);J. Phys. **A 34**, 2923 (2001).

[80] J. P. Eckmann, J. Magnen, and R. Sénéor, Comm. Math. Phys. **39**, 251 (1975); J. Magnen and R. Sénéor, Comm. Math. Phys. **56**, 237 (1977); J. S. Feldman and K. Osterwalder, Ann. Phys. (NY) **97**, 80 (1976); J. Magnen and V. Rivasseau, Comm. Math. Phys. **102**, 59 (1985).

[81] J. C. Le Guillou and J. Zinn-Justin, Phys. Rev. Lett. **39**, 95 (1977); Phys. Rev. **B 21**, 3976 (1980).

[82] L. N. Lipatov, Zh. Eksp. Teor. Fiz. **72**, 411 (1977) [Sov. Phys. JETP **45**, 216 (1977)].

[83] E. Brézin, J. C. Le Guillou, and J. Zinn-Justin, Phys. Rev. **D 15**, 1544, 1558 (1977).

[84] J. C. Le Guillou and J. Zinn-Justin, J. Physique Lett. **46**, L137 (1985); J. Physique **48**, 19 (1987).

[85] R. Guida and J. Zinn-Justin, J. Phys. **A 31**, 8103 (1998).

[86] P. Calabrese and P. Parruccini, Phys. Rev. **B 64**, 184408 (2001).

[87] A. J. Bray, T. McCarthy, M. A. Moore, J. D. Reger, and A. P. Young, Phys. Rev. **B 36**, 2212 (1987); A. J.

McKane, Phys. Rev. **B 49**, 12003 (1994);

[88] G. Alvarez, V. Martín-Mayor, and J. J. Ruiz-Lorenzo, J. Phys. **A 33**, 841 (2000).

[89] P. Calabrese, M. Caselle, A. Celi, A. Pelissetto, and E. Vicari, J. Phys. A **33**, 8155 (2000).

[90] B. G. Nickel, in *Phase Transitions*, M. Lévy, J. C. Le Guillou, and J. Zinn-Justin eds., (Plenum, New York and London, 1982).

[91] B. G. Nickel, Physica **A 117**, 189 (1991).

[92] A. Pelissetto and E. Vicari, Nucl. Phys. **B 519**, 626 (1998); Nucl. Phys. B (Proc. Suppl.) **73**, 775 (1999).

[93] A. Pelissetto and E. Vicari, Nucl. Phys. B **575**, 579 (2000).

[94] J. Balog, M. Niedermaier, F. Niedermayer, A. Patrascioiu, E. Seiler, and P. Weisz Phys. Rev. **D 60** 094508 (1999); Nucl. Phys. B **583**, 614 (2000); "Does the XY model have an integrable continuum limit?", e-print hep-lat/0106015

[95] J. Kim, Phys. Lett. B **345**, 469 (1995).

[96] In order to find more precise estimates of the cubic fixed point for $N \geq 3$ we also perform a constrained analysis of the function $\bar{\beta}_{\bar{v}}$ fixing the zero for $\bar{u} = 0$ at $\bar{v} = \bar{g}_I^*$. This analysis brings to results substantially equivalent to the ones of Table VI.

[97] In order to relate the physical anisotropy to the experimentally measured exponents, it remains to understand the relation between the ratio of the bare parameter u_0 and v_0 (that are the physical couplings) and their fixed point values. This could be obtained by a Monte Carlo simulation of the Hamiltonian (1).

[98] A. Aharony, Phys. Rev. **B 8**, 3349 (1973).

[99] A. D. Bruce, J. Phys. **C 7**, 2089 (1974).

[100] T. Nattermann and S. Trimper, J. Phys. **A 8**, 2000 (1975).

[101] A. D. Bruce and A. Aharony, Phys. Rev. **B 11**, 478 (1975).

[102] S. Elitzur, R. B. Pearson, and J. Shigemitsu, Phys. Rev. **D 19**, 3698 (1979); B. Bonnier, M. Hontebeyrie, and C. Meyers Phys. Rev. **B 39**, 4079 (1989).